

# THE FREE ENTROPY DIMENSION OF SOME $L^\infty[0, 1]$ -CIRCULAR OPERATORS

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ABSTRACT. We find the microstates free entropy dimension of a large class of  $L^\infty[0, 1]$ -circular operators, in the presence of a generator of the diagonal subalgebra.

## 1. INTRODUCTION

Let  $\mathcal{M}$  be a von Neumann algebra with a specified normal faithful tracial state  $\tau$ . The free entropy dimension

$$\delta_0(X_1, \dots, X_n) \quad (1.1)$$

for  $X_1, \dots, X_n \in \mathcal{M}$ , was introduced by Voiculescu [21], [22], see also [25]. This quantity is sometimes called the microstates free entropy dimension to distinguish it from another version introduced by Voiculescu and because its definition utilizes matricial microstates for the operators  $X_1, \dots, X_n$ . It is an open problem whether the quantity (1.1) is an invariant of the von Neumann algebra generated by  $X_1, \dots, X_n$ , and it is of interest to find the free entropy dimension of various operators. See, for example [22], [24] [7], [8], [6], [10], [11], [12], [14] for some such results.

In [6], Dykema, Jung and Shlyakhtenko computed  $\delta_0(T) = 2$  for the quasiniipotent DT-operator  $T$ . This operator was introduced by Dykema and Haagerup in [4]. It can be realized as a limit in  $*$ -moments of strictly upper-triangular random matrices with i.i.d. complex Gaussian entries above the diagonal. Alternatively, as was seen in [4],  $T$  can be obtained in the free group factor  $L(\mathbb{F}_2)$  from a semicircular element  $X$  and a free copy of  $L^\infty([0, 1])$  by using projections from the latter to cut out the upper triangular part of  $X$ . (Note that  $X$  may be replaced by a circular element  $Z$  for this procedure.) Then we can visualize  $T$  as in Figure 1, where the shaded region has weight 1, the unshaded region has weight 0, and these weights are used to multiply entries of a Gaussian random matrix. It was proved in [5] that the von Neumann algebra generated by  $T$  contains all of  $L^\infty([0, 1])$ , and is, thus, the free group factor  $L(\mathbb{F}_2)$ .

In this paper we consider more general operators than  $T$ , defined also as limits of random matrices or, equivalently, in the approach was taken in [3], by cutting a circular operator  $Z$  using projections in a  $*$ -free copy of  $L^\infty([0, 1])$ . The class of operators considered there consisted of those  $L^\infty([0, 1])$ -circular operators described as follows. Let  $\eta$  be an absolutely continuous measure with respect to Lebesgue measure on  $[0, 1]^2$  with Radon–Nikodym derivative  $H \in L^1([0, 1]^2)$  and assume the push–forward measures  $\pi_{i*}\eta$  under the coordinate projections  $\pi_1, \pi_2 : [0, 1]^2 \rightarrow [0, 1]$  are absolutely continuous with respect to Lebesgue measure and have essentially bounded Radon–Nikodym derivatives. For each such measure  $\eta$  with the associated

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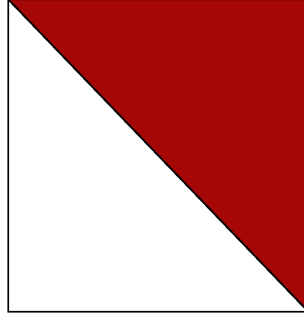


FIGURE 1. The upper triangle, representing the quasinilpotent DT-operator  $T$

function  $H \in L^1([0, 1]^2)$  we have the operator  $Z_H$  described in [3]; (however, this operator was denoted  $z_\eta$  in [3]). When  $\eta$  is Lebesgue measure on  $[0, 1]^2$ , then  $H = 1$  and  $Z_H$  is the usual circular operator. When  $\eta$  is the restriction of Lebesgue measure to the upper triangle pictured in Figure 1, then  $H$  is the characteristic function of this triangle and  $Z_H$  is the quasinilpotent DT-operator  $T$ .

Let  $D \in L^\infty([0, 1])$  be the identity map from  $[0, 1]$  to itself; thus,  $D$  generates  $L^\infty([0, 1])$ . In this paper, with  $H$  as above, we compute the free entropy dimension  $\delta_0(Z_H : D)$  of  $Z_H$  in the presence of  $D$ , in the case  $H$  satisfies certain additional hypothesis, showing that then

$$\delta_0(Z_H : D) = 1 + 2 \text{ area}(\text{supp}(H)), \quad (1.2)$$

where  $\text{supp}(H)$  is the measurable support of  $H$  and where the area is Lebesgue measure. We prove the upper bound  $\leq$  in (1.2) for general  $H$ , (see Theorem 3.3) using basic estimates inspired by [26]. We prove the lower bound  $\geq$  in (1.2) for all  $H$  that are supported in the upper triangle as drawn in Figure 1 and whose restrictions to some band as drawn in Figure 2 are nonzero constant. (Actually, somewhat weaker conditions suffice — see Theorem 3.2.) Our proof of the lower bound uses techniques similar to those used in [6].

The organization of the rest of this paper is as follows. In §2, we discuss some definitions and results that we need for the calculation. These include (§2.1) basic facts about the class of  $L^\infty([0, 1])$ -circular operators that we consider, their construction in  $L(\mathbb{F}_2)$  and a lemma about them; (§2.2) a result about certain matrix approximants to the quasinilpotent DT-operator which was lifted from [6] but that follows directly from work of Aagaard and Haagerup [1] and Śniady [17]; (§2.3) Jung's equivalent approach to free entropy dimension in terms of packing numbers [9]; (§2.4) Dyson's formula for the volumes of sets of matrices that are invariant under unitary conjugation. In §3, we prove the main result, namely the equation (1.2). Finally, in §4, we consider an example when  $\delta_0(Z_H : D) < \delta_0(Z_H)$  and we ask a natural question. *Acknowledgement:* The first named author thanks Kenley Jung for helpful comments.

## 2. DEFINITIONS AND PRELIMINARIES

**2.1.  $L^\infty([0, 1])$ -circular operators in free group factors.** In this section we recall how  $L^\infty([0, 1])$ -circular operators in a certain class were constructed in [3], and we

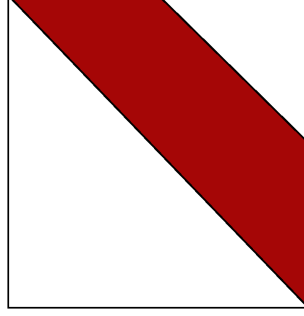


FIGURE 2. A band above the diagonal

prove a lemma. We work in  $W^*$ -noncommutative probability space  $(\mathcal{M}, \tau)$ , with  $\tau$  a faithful trace, and we fix a copy  $\mathcal{A} = L^\infty[0, 1] \subseteq \mathcal{M}$ , such that the restriction of  $\tau$  to  $\mathcal{A}$  is given by integration with respect to Lebesgue measure on  $[0, 1]$ . Let  $D \in \mathcal{A}$  be the operator corresponding the function in  $L^\infty[0, 1]$  that is the identity map from  $[0, 1]$  to itself. Let  $E : \mathcal{M} \rightarrow \mathcal{A}$  be the  $\tau$ -preserving conditional expectation. Let  $H \in L^1([0, 1]^2)$ ,  $H \geq 0$ , and assume  $H$  has essentially bounded coordinate expectations  $CE_1(H)$  and  $CE_2(H)$ , given by

$$CE_1(H)(x) = \int_0^1 H(x, y) dy, \quad CE_2(H)(y) = \int_0^1 H(x, y) dx. \quad (2.1)$$

By  $Z_H$ , we will denote an  $\mathcal{A}$ -circular operator in  $(\mathcal{M}, E)$  with covariance  $(\alpha_H, \beta_H)$  where  $\alpha_H, \beta_H : L^\infty[0, 1] \rightarrow L^\infty[0, 1]$  are given by

$$\alpha_H(f)(x) = \int_0^1 H(t, x) f(t) dt, \quad \beta_H(f)(x) = \int_0^1 H(x, t) f(t) dt. \quad (2.2)$$

Suppose  $Z \in \mathcal{M}$  is a  $(0, 1)$ -circular element, namely a circular element satisfying  $\tau(Z) = 0$  and  $\tau(Z^*Z) = 1$ , and suppose  $\mathcal{A}$  and  $\{Z\}$  are  $*$ -free. We will construct our operator  $Z_H$  from  $\mathcal{A}$  and  $Z$  as in Theorem 6.5 of [3]. (Note that our notation differs slightly from that used in [3].)

**Definition 2.1.** Let  $\omega \in L^\infty([0, 1]^2)$ . We say that  $\omega$  is in regular block form if  $\omega$  is constant on all blocks in the regular  $n \times n$  lattice superimposed on  $[0, 1]^2$ , for some  $n$ , i.e. if there are  $n \in \mathbb{N}$  and  $\omega_{i,j} \in \mathbb{C}$ ,  $(1 \leq i, j \leq n)$  such that  $\omega(s, t) = \omega_{i,j}$  whenever  $\frac{i-1}{n} \leq s \leq \frac{i}{n}$  and  $\frac{j-1}{n} \leq t \leq \frac{j}{n}$ , for all integers  $1 \leq i, j \leq n$ . (We then say  $\omega$  is in  $n \times n$  regular block form.) Then we set

$$M(\omega, Z) = \sum_{i,j=1}^n \omega_{i,j} p_i Z p_j$$

where  $p_i = \mathbf{1}_{[\frac{i-1}{n}, \frac{i}{n}]} \in \mathcal{A}$ . Note that we have  $M(\omega, Z) \in W^*(\mathcal{A} \cup \{Z\}) \cong L(\mathbb{F}_3)$ .

Recalling Lemma 6.4 and Theorem 6.5 of [3] we can state the following theorem.

**Theorem 2.2.** Let  $\omega = \sqrt{H}$ . Then there exists a sequence  $\{\omega^{(n)}\}_n$  in  $L^\infty([0, 1]^2)$  such that

- (i) for each  $n$ ,  $\omega^{(n)}$  is in regular block form,
- (ii)  $\lim_n \|\omega - \omega^{(n)}\|_{L^2} = 0$

- (iii) letting  $H^{(n)} = (\omega^{(n)})^2$ , both  $\|CE_1(H^{(n)})\|_\infty$  and  $\|CE_2(H^{(n)})\|_\infty$  remain bounded as  $n$  goes to  $\infty$ .

Moreover, there is an  $L^\infty[0, 1]$ -circular operator  $Z_H$  with covariance  $(\alpha_H, \beta_H)$  as described in equations (2.2) such that whenever  $\{\omega^{(n)}\}_n$  is a sequence satisfying conditions (i)–(iii) above, the operators  $M(\omega^{(n)}, Z)$  as given in Definition 2.1 converge in the strong-operator-topology as  $n \rightarrow \infty$  to  $Z_H$ .

**Remark 2.3.** Of particular interest is the operator  $Z_R$  when  $R = 1_{\{(s,t)|s < t\}}$  is the characteristic function in the upper triangle in  $[0, 1]^2$ . This  $Z_R$  is an instance of the  $DT(\delta_0, 1)$ -operator, also called the quasinilpotent  $DT$ -operator, and also denoted  $T$ . The construction of  $Z_R$  in Theorem 2.2 above is approximately what was done in §4 of [4].

The following lemma will be used in §3 to prove the upper bound on free entropy dimension. For emphasis, we will denote by  $\lambda : L^\infty[0, 1] \rightarrow \mathcal{M}$  the identification of  $L^\infty[0, 1]$  (with its trace given by Lebesgue measure) and  $\mathcal{A} = \lambda(L^\infty[0, 1]) \subseteq \mathcal{M}$ .

**Lemma 2.4.** Let  $T = Z_R \in W^*(\{Z\} \cup \mathcal{A})$  be the quasinilpotent  $DT$ -operator as described in Remark 2.3. Let  $N$  be an integer,  $N \geq 2$ . Assume for all  $i, j \in \{1, \dots, N\}$  with  $i \neq j$ ,  $Y_{i,j} \in \mathcal{M}$  is a  $(0, 1)$ -circular element such that the family

$$\mathcal{A}, \quad \{Z\}, \quad (\{Y_{i,j}\})_{1 \leq i, j \leq N, i \neq j}$$

is  $*$ -free. Let  $(e_{ij})_{1 \leq i, j \leq N}$  be a system of matrix units for  $M_N(\mathbb{C})$ . Consider the  $*$ -noncommutative probability space  $(\mathcal{M} \otimes M_N(\mathbb{C}), \tau \otimes \text{tr}_N)$ , and let  $\tilde{\lambda} : L^\infty[0, 1] \rightarrow \mathcal{M} \otimes M_N(\mathbb{C})$  be the  $*$ -homomorphism given by

$$\tilde{\lambda}(f) = \sum_{j=1}^N \lambda(f \circ \rho_j) \otimes e_{jj},$$

where  $\rho_j : [0, 1] \rightarrow [0, 1]$  is  $\rho_j(t) = \frac{t}{N} + \frac{j-1}{N}$ . Let  $\tilde{\mathcal{A}} = \tilde{\lambda}(L^\infty[0, 1])$ . Then the  $\tau \otimes \text{tr}_N$ -preserving conditional expectation  $\tilde{E} : \mathcal{M} \otimes M_N(\mathbb{C}) \rightarrow \tilde{\mathcal{A}}$  is given by

$$\tilde{E}\left(\sum_{1 \leq i, j \leq N} a_{ij} \otimes e_{ij}\right) = \sum_{j=1}^N E(a_{jj}) \otimes e_{jj}.$$

Let  $c_{ij} \in [0, \infty)$  ( $1 \leq i, j \leq N$ ,  $i \neq j$ ) and let

$$\tilde{Y} = \frac{1}{\sqrt{N}} \left( \sum_{k=1}^N T \otimes e_{kk} + \sum_{1 \leq i, j \leq N, i \neq j} c_{ij} Y_{ij} \otimes e_{ij} \right).$$

Then  $\tilde{Y}$  is  $\tilde{\mathcal{A}}$ -circular with covariance  $(\alpha_H, \beta_H)$  as given in (2.2), where

$$H(s, t) = \begin{cases} 1, & \frac{k-1}{N} \leq s \leq t \leq \frac{k}{N}, 1 \leq k \leq N \\ (c_{ij})^2, & \frac{i-1}{N} \leq s \leq \frac{i}{N}, \frac{j-1}{N} \leq t \leq \frac{j}{N}, 1 \leq i, j \leq N, i \neq j. \end{cases}$$

*Proof.* Let

$$\tilde{Z} = \frac{1}{\sqrt{N}} \left( \sum_{k=1}^N Z \otimes e_{kk} + \sum_{1 \leq i, j \leq N, i \neq j} Y_{ij} \otimes e_{ij} \right).$$

We will show that  $\tilde{Z}$  is  $(0, 1)$ -circular and is  $*$ -free from  $\tilde{\mathcal{A}}$ . Let  $u_1, \dots, u_N \in \mathcal{M}$  be Haar unitary elements such that the family

$$(\{u_k, u_k^*\})_{1 \leq k \leq N}, \quad \mathcal{A}, \quad \{Z\}, \quad (\{Y_{i,j}\})_{1 \leq i,j \leq N, i \neq j}$$

is  $*$ -free (after enlarging  $(\mathcal{M}, \tau)$  if necessary). Let

$$U = \sum_{k=1}^N u_k \otimes e_{kk}.$$

It will suffice to show that  $U^* \tilde{Z} U$  is  $(0, 1)$ -circular and is  $*$ -free from  $U^* \tilde{\mathcal{A}} U$ . For this, by results following directly from Voiculescu's matrix model [20] (see [19]), it will suffice to show that each  $u_k^* Z u_k$  and each  $u_i^* Y_{ij} u_j$  is circular and that the family

$$(\{u_k^* Z u_k\})_{1 \leq k \leq N}, \quad (\{u_i^* Y_{ij} u_j\})_{1 \leq i,j \leq N, i \neq j}, \quad (u_k^* \mathcal{A} u_k)_{1 \leq k \leq N} \quad (2.3)$$

is  $*$ -free in  $(\mathcal{M}, \tau)$ . Let  $Z = V|Z|$  and  $Y_{ij} = V_{ij}|Y_{ij}|$  be the polar decompositions. Then (see [19]),  $V$  and  $V_{ij}$  are Haar unitaries,  $|Z|$  and  $|Y_{ij}|$  are quarter-circular elements,  $V$  and  $|Z|$  are  $*$ -free and, for each  $i$  and  $j$ ,  $V_{ij}$  and  $|Y_{ij}|$  are  $*$ -free in  $(\mathcal{M}, \tau)$ . We have the polar decompositions

$$\begin{aligned} u_k^* Z u_k &= (u_k^* V u_k)(u_k^* |Z| u_k) \\ u_i^* Y_{ij} u_j &= (u_i^* V_{ij} u_j)(u_i^* |Y_{ij}| u_j). \end{aligned}$$

Therefore, in order to show that  $*$ -freeness of the family (2.3) and circularity of  $u_k^* Z u_k$  and  $u_i^* Y_{ij} u_j$ , it will suffice to show  $*$ -freeness of the family

$$\begin{aligned} &(\{u_k^* |Z| u_k\})_{1 \leq k \leq N}, \quad (\{u_k^* V u_k\})_{1 \leq k \leq N}, \\ &(\{u_i^* |Y_{ij}| u_j\})_{1 \leq i,j \leq N, i \neq j}, \quad (\{u_i^* V_{ij} u_j\})_{1 \leq i,j \leq N, i \neq j}, \quad (u_k^* \mathcal{A} u_k)_{1 \leq k \leq N}. \end{aligned}$$

Let  $B$  be a Haar unitary generating  $W^*(|Z|)$ , let  $B_{ij}$  be a Haar unitary generating  $W^*(|Y_{ij}|)$ , and let  $C$  be a Haar unitary generating  $\mathcal{A}$ . It will suffice to show  $*$ -freeness of the family

$$\begin{aligned} &(u_k^* B u_k)_{1 \leq k \leq N}, \quad (u_k^* V u_k)_{1 \leq k \leq N}, \\ &(u_j^* B_{ij} u_j)_{1 \leq i,j \leq N, i \neq j}, \quad (u_i^* V_{ij} u_j)_{1 \leq i,j \leq N, i \neq j}, \quad (u_k^* C u_k)_{1 \leq k \leq N} \end{aligned}$$

of Haar unitaries. This follows from the  $*$ -freeness of the family

$$B, C, V, (u_k)_{1 \leq k \leq N}, \quad (B_{ij})_{1 \leq i,j \leq N, i \neq j}, \quad (V_{ij})_{1 \leq i,j \leq N, i \neq j}.$$

by an argument involving words in a free group. This shows that  $\tilde{Z}$  is  $(0, 1)$ -circular and  $*$ -free from  $\tilde{\mathcal{A}}$ .

Now we use the method of Theorem 6.5 of [3], described in Theorem 2.2 above, but taking  $\omega^{(n)}$  in  $n \times n$  regular block form with  $n$  always a multiple of  $N$ , and with each such  $\omega^{(n)}$  constant equal to  $c_{ij}$  on each off-diagonal block of the form  $[\frac{i-1}{N}, \frac{i}{N}] \times [\frac{j-1}{N}, \frac{j}{N}]$  for  $1 \leq i, j \leq N, i \neq j$ , where projections from  $\tilde{\mathcal{A}}$  are used to cut  $\tilde{Z}$  and make each  $M(\omega^{(n)}, \tilde{Z})$ . It is then clear that the operators  $M(\omega^{(n)}, \tilde{Z})$  converge to  $\tilde{Y}$  as  $n \rightarrow \infty$ , and, from Theorem 2.2, they also converge to an  $\tilde{\mathcal{A}}$ -circular operator having the desired covariance  $(\alpha_H, \beta_H)$ .  $\square$

**2.2. Microstates for the quasinilpotent DT-operator.** Let  $T = Z_R$  be the quasinilpotent DT-operator as described in Remark 2.3 and let  $D$  be the corresponding operator described in §2.1. It was proved by Aagard and Haagerup [1] that if we consider  $T$  a  $\text{DT}(\delta_0, 1)$ -operator and  $Y$  a circular operator that is  $*$ -free from  $T$  (and  $D$ ), then the Brown measure of  $T + \epsilon Y$  is equal to the uniform distribution on the closed disk centered at 0 and of radius  $r_\epsilon = \log(1 + \epsilon^{-2})^{-\frac{1}{2}}$ . Note how slowly this disk shrinks as  $\epsilon$  approaches to 0. Moreover, they also showed that the spectrum of  $T + \epsilon Y$  is equal to the disk.

The next lemma is an immediate consequence of the above described Brown measure result of Aagaard and Haagerup and a result of Śniady [17]. A detailed proof can be formulated exactly as was done for Lemma 2.2 in [6]. In the following lemma and throughout this paper, for a matrix  $A \in M_k(\mathbb{C})$  we let  $|A|_2 = \text{tr}_k(A^*A)^{1/2}$ , where  $\text{tr}_k$  is the normalized trace of  $M_k(\mathbb{C})$ . Also, by the eigenvalue distribution of a matrix  $A \in M_k(\mathbb{C})$  we mean the probability measure  $\frac{1}{n} \sum_1^n \delta_{\lambda_j}$ , where  $\lambda_1, \dots, \lambda_k$  are the eigenvalues of  $A$  listed according to general multiplicity.

**Lemma 2.5.** *Let  $c > 0$ . Then there exists sequences  $\{g_k\}_k$  and  $\{y_k\}_k$  such that for any  $\epsilon > 0$ , there exists a sequence  $\{z_{k,\epsilon}\}_k$  such that*

- $g_k, y_k, z_{k,\epsilon} \in M_k(\mathbb{C})$ ,
- $\|g_k\|, \|y_k\|$  and  $\|z_{k,\epsilon}\|$  remain bounded as  $k \rightarrow +\infty$ ,
- $\limsup_k |y_k - z_{k,\epsilon}|_2 \leq \epsilon c$ ,
- the pair  $(g_k, y_k)$  converges in  $*$ -moments as  $k \rightarrow +\infty$  to the pair  $(D, T)$ ,
- the eigenvalue distribution of  $z_{k,\epsilon}$  converges weakly as  $k \rightarrow +\infty$  to the measure  $\sigma_{\epsilon,c}$ , which is the uniformly distributed measure in the disk of center at 0 and radius  $r_{\epsilon,c} = c \log(1 + \epsilon^{-2})^{-\frac{1}{2}}$  in the complex plane.

**2.3. Packing number formulation of the free entropy dimension.** In this section we will review the packing number formulation of Voiculescu's microstates free entropy dimension due to K. Jung [9]. If  $X = (x_1, \dots, x_n)$  and  $Z = (z_1, \dots, z_m)$  are tuples of selfadjoint elements in a tracial von Neumann algebra, then the microstates free entropy dimension (as defined by Voiculescu [22]) is given by the formula

$$\delta_0(X) = n + \limsup_{\epsilon \rightarrow 0} \frac{\chi(x_1 + \epsilon s_1, \dots, x_n + \epsilon s_n : s_1, \dots, s_n)}{|\log \epsilon|}$$

and the microstates free entropy dimension in the presence of  $Z$  is defined by

$$\delta_0(X : Z) = n + \limsup_{\epsilon \rightarrow 0} \frac{\chi(x_1 + \epsilon s_1, \dots, x_n + \epsilon s_n : z_1, \dots, z_m, s_1, \dots, s_n)}{|\log \epsilon|}$$

where  $\{s_1, \dots, s_n\}$  is a semicircular family free from  $X$  and  $Z$ . The packing formulation found in [9] is

$$\delta_0(X) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} \quad \delta_0(X : Z) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(X : Z)}{|\log \epsilon|} \quad (2.4)$$

where

$$\mathbb{P}_\epsilon(X) = \inf_{m,\gamma} \limsup_k k^{-2} \log P_\epsilon(\Gamma(X; m, k, \gamma))$$

and

$$\mathbb{P}_\epsilon(X : Z) = \inf_{m,\gamma} \limsup_k k^{-2} \log P_\epsilon(\Gamma(X : Z; m, k, \gamma))$$

Here,  $\Gamma(X : Z; m, k, \gamma) \subseteq (M_k(\mathbb{C})_{s.a.})^n$  is the microstates space of Voiculescu, and  $P_\epsilon$  is the packing number with respect to the metric arising from the normalized trace. Let  $Y = (y_1, \dots, y_n)$  and  $W = (w_1, \dots, w_m)$  be arbitrary tuples of possibly non-selfadjoint elements in a tracial von Neumann algebra. Now the definition of  $\mathbb{P}_\epsilon$  makes perfect sense for the set  $Y$  if we replace the microstates space in (2.4) with the non-selfadjoint  $*$ -microstates space  $\Gamma(Y : W; m, k, \gamma) \subseteq (M_k(\mathbb{C}))^n$ , which is the set of all  $n$ -tuples of  $k \times k$  matrices whose  $*$ -moments up to order  $m$  approximate those of  $Y$  within tolerance of  $\gamma$  in the presence of  $W$ . It is also true that

$$\delta_0(\operatorname{Re}(y_1), \operatorname{Im}(y_1), \dots, \operatorname{Re}(y_n), \operatorname{Im}(y_n) : W) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(Y : W)}{|\log \epsilon|}$$

see [6] for details.

Finally, we review the standard volume comparison inequality for packing numbers. Recall that for a metric space  $A$  we have

$$P_{4\epsilon}(A) \leq K_{2\epsilon}(A) \leq P_\epsilon(A),$$

where  $P_\epsilon(A)$  is the  $\epsilon$ -packing number, i.e. the maximal number of disjoint open balls of radius  $\epsilon$  in  $A$ , and  $K_\epsilon(A)$  is the minimal number of elements in a cover of  $A$  consisting of open balls of radius  $\epsilon$ . If  $A$  is a subspace of a Euclidean space, then we have

$$\operatorname{vol}(\mathcal{N}_\epsilon(A)) \leq K_\epsilon(A) \cdot \operatorname{vol}(\mathcal{B}_{2\epsilon}),$$

where  $\mathcal{N}_\epsilon(A)$  is the  $\epsilon$ -neighborhood,  $\mathcal{B}_r$  is a ball of radius  $r$  and  $\operatorname{vol}$  is the volume, all in the ambient Euclidean space. We thus have the volume comparison test,

$$P_\epsilon(A) \geq K_{2\epsilon}(A) \geq \frac{\operatorname{vol}(\mathcal{N}_{2\epsilon}(A))}{\operatorname{vol}(\mathcal{B}_{4\epsilon})}. \quad (2.5)$$

**2.4. Dyson's formula.** Every matrix of  $M_k(\mathbb{C})$  has an upper-triangular matrix in its unitary orbit. Thus, letting  $T_k(\mathbb{C})$  denote the set of upper-triangular matrices in  $M_k(\mathbb{C})$ , there is a probability measure  $\nu_k$  on  $T_k(\mathbb{C})$  such that

$$\lambda_k(\mathcal{O}) = \nu_k(\mathcal{O} \cap T_k)$$

for every  $\mathcal{O} \subseteq M_k(\mathbb{C})$  that is invariant under unitary conjugation. Freeman Dyson identified such a measure [15], and showed that if we view  $T_k(\mathbb{C})$  as a Euclidean space of real dimension  $k(k+1)$  with coordinates corresponding to the real and imaginary part of the matrix entries lying on and above the diagonal, then  $\nu_k$  is absolutely continuous with respect to Lebesgue measure on  $T_k(\mathbb{C})$  and has density given at  $A = (a_{ij})_{1 \leq i, j \leq k} \in T_k(\mathbb{C})$  by

$$C_k \cdot \prod_{1 \leq p < q \leq k} |a_{pp} - a_{qq}|^2 \quad \text{where} \quad C_k = \frac{\pi^{k(k+1)/2}}{\prod_{j=1}^k j!}. \quad (2.6)$$

We will use Dyson's formula in our main result to find lower bound on the volume of unitary orbits of an  $\epsilon$ -neighborhood of the microstates space.

## 3. FREE ENTROPY DIMENSION COMPUTATIONS

**Lemma 3.1.** *Let  $(\Omega, \mu)$  a finite measurable space. Let  $f \in L^1(\Omega)$  and  $f \geq 0$ . Then*

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} \log(\max(f(t), \epsilon)) d\mu(t)}{|\log \epsilon|} = \mu(\text{supp}(f)) - \mu(\Omega),$$

where  $\text{supp}(f) = f^{-1}((0, +\infty))$ .

*Proof.* It is clear that we have  $\log(\max(f(t), \epsilon)) \leq \log(f(t) + 1) + \log(\epsilon) \cdot \mathbf{1}_{f^{-1}([0, \epsilon))}$ , and this yields

$$\limsup_{\epsilon \rightarrow 0} \frac{\int_{\Omega} \log(\max(f(t), \epsilon)) d\mu(t)}{|\log \epsilon|} \leq -\liminf_{\epsilon \rightarrow 0} \mu(f^{-1}([0, \epsilon))) = \mu(\text{supp}(f)) - \mu(\Omega).$$

On the other hand, given  $\gamma > 0$ , let  $\delta > 0$  be such that  $\mu(f^{-1}((0, \delta))) < \gamma$ . Taking  $0 < \epsilon < \delta$ , we have  $\mathbf{1}_{f^{-1}([0, \delta))} \cdot \log \epsilon + \mathbf{1}_{f^{-1}([\delta, +\infty))} \cdot \log \delta \leq \log \max(f(t), \epsilon)$  and integrating on both sides we obtain

$$\mu(f^{-1}([0, \delta))) \cdot \log \epsilon + \mu(f^{-1}([\delta, +\infty))) \cdot \log \delta \leq \int_{\Omega} \log(\max(f(t), \epsilon)) d\mu(t).$$

Now dividing by  $|\log \epsilon|$  and taking  $\liminf$  on both sides we get

$$-\mu(f^{-1}([0, \delta))) \leq \liminf_{\epsilon \rightarrow 0} \frac{\int_{\Omega} \log(\max(f(t), \epsilon)) d\mu(t)}{|\log \epsilon|}.$$

Using the fact that  $\mu(f^{-1}([0, \delta))) < \mu(f^{-1}(0)) + \gamma$  and that  $\gamma$  is arbitrary we obtain

$$\mu(\text{supp}(f)) - \mu(\Omega) \leq \liminf_{\epsilon \rightarrow 0} \frac{\int_{\Omega} \log(\max(f(t), \epsilon)) d\mu(t)}{|\log \epsilon|},$$

proving the claim.  $\square$

As in §2.1, we work in  $(\mathcal{M}, \tau)$  and we have  $\mathcal{A} = L^\infty[0, 1]$  and a  $(0, 1)$ -circular element  $Z$  such that  $\mathcal{A}$  and  $Z$  are  $*$ -free, and with  $H$  as described there. We construct as in §2.1 an  $L^\infty[0, 1]$ -circular operator  $Z_H \in W^*(\mathcal{A} \cup \{Z\}) \cong L(\mathbb{F}_3)$ . We also take  $D = D^* \in \mathcal{A}$  to correspond to the identity function from  $[0, 1]$  to itself. The following is our main result.

**Theorem 3.2.** *Let  $H \geq 0$ ,  $H \in L^1([0, 1]^2)$  have essentially bounded coordinate expectations  $CE_1(H)$  and  $CE_2(H)$ , as in equations (2.1). Assume  $H$  has support contained in the upper-triangle  $U$  of  $[0, 1]^2$  and assume there exists  $r \in \mathbb{N}$  such that*

$$\Delta := \bigcup_{i=1}^r U_i^{(r)} \subseteq \text{supp}(H), \quad U_i^{(r)} = \{(x, y) : \frac{i-1}{r} \leq x < y \leq \frac{i}{r}\}$$

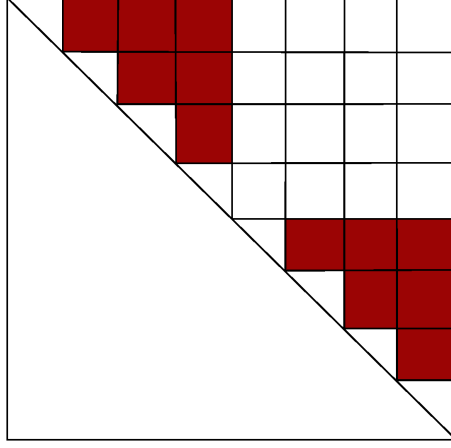
and that  $H$  restricted to  $\Delta$  is constant equal to  $c > 0$ . Then

$$\delta_0(Z_H : D) \geq 1 + 2 \cdot \text{area}(\text{supp}(H)).$$

In particular,  $\delta_0(Z_H) \geq 1 + 2 \cdot \text{area}(\text{supp}(H))$ .

*Proof.* Without loss of generality we can assume  $c = 1$ . Fix  $\epsilon > 0$ . By hypothesis we may choose  $N$  arbitrarily large and so that  $\bigcup_{i=1}^N U_i^{(N)} \subseteq \Delta$ . Let  $R > 1$ ,  $m \in \mathbb{N}$



FIGURE 3. Case  $N = 2$  and  $p = 4$ 

and  $\gamma > 0$ . There is  $\delta > 0$  such that  $\|Z_H - Y\|_2 < \delta$  implies  $\Gamma_R(Y; m, k, \gamma/2) \subseteq \Gamma_R(Z_H; m, k, \gamma)$ . Making use of Theorem 2.2, there exist  $M = Np$  and

$$\omega := \sum_{i=1}^M \mathbf{1}_{U_i^{(M)}} + \sum_{1 \leq i < j \leq M} \alpha_{ij} \mathbf{1}_{E_{ij}^{(M)}}$$

where  $E_{ij}^{(M)} = \{(x, y) : \frac{i-1}{M} \leq x \leq \frac{i}{M}, \frac{j-1}{M} \leq y \leq \frac{j}{M}\}$  with  $\alpha_{ij} > 0$ , such that  $\|Z_H - Z_\omega\|_2 < \delta$  and, therefore, we have  $\Gamma_R(Z_\omega; m, k, \gamma/2) \subseteq \Gamma_R(Z_H; m, k, \gamma)$ . We define the sets of indices

$$\Theta = \{(i, j) : 1 \leq i < j \leq p, p+1 \leq i < j \leq 2p, \dots, (N-1)p+1 \leq i < j \leq Np\}$$

and

$$\Phi = \{(i, j) : 1 \leq i < j \leq Np\} \setminus \Theta.$$

For example, in the case  $N = 2$  and  $p = 4$  the squares corresponding to elements of  $\Theta$  are shaded in Figure 3. Note that by the hypothesis of  $H$  we may insist,  $\alpha_{ij} = 1$  whenever  $(i, j) \in \Theta$ . Let  $\gamma' = \gamma/(MR)^{m-1}$ .

Consider  $(C_{11}, \dots, C_{MM})$ ,  $(C_{ij})_{1 \leq i < j \leq M}$  a  $*$ -free family in  $(\mathcal{M}, \tau)$ , where each  $C_{ii}$  is  $\text{DT}(\delta_0, \frac{1}{\sqrt{M}})$ , and each  $C_{ij}$  with  $i < j$  is circular with  $\tau(|C_{ij}^2|) = \frac{1}{M}$ . Let  $\{g_k\}_k$  and  $\{y_k\}_k$  the sequences constructed in Lemma 2.5 with  $c = 1/\sqrt{M}$ . There are  $a_{ij}(k) \in M_k(\mathbb{C})$  for  $(i, j) \in \Theta$  such that for each  $(i, j) \in \Theta$  as before  $a_{ij}(k)$  converge in distribution as  $k \rightarrow +\infty$  to a  $(0, \frac{1}{M})$ -circular element and such that the family

$$\{g(k), y(k)\}, (\{a_{ij}(k)\})_{(i,j) \in \Theta}$$

of sets of random variables is asymptotically  $*$ -free as  $k \rightarrow \infty$ . By an application of Corollary 2.14 of [23], for  $k$  large enough there exists a set  $\Omega_k \subset \Gamma((C_{ij})_{(i,j) \in \Phi}; m, k, \gamma')$  such that for any  $(\eta_{ij})_{(i,j) \in \Phi} \in \Omega_k$ ,

$$\{y_k, g(k)\}, (a_{ij}(k))_{(i,j) \in \Theta}, (\eta_{ij})_{(i,j) \in \Phi}$$

is an  $(m, \gamma')$ -free family of sets of random variables and

$$\begin{aligned} \liminf_k \left( k^{-2} \cdot \log(\text{vol}(\Omega_k)) + \left( \frac{N(N-1)p^2}{2} \right) \cdot \log(k) \right) &\geq \\ &\geq \chi((\text{Re } C_{ij})_{(i,j) \in \Phi}, (\text{Im } C_{ij})_{(i,j) \in \Phi}) > -\infty \end{aligned} \quad (3.1)$$

where the volume is computed with respect to the Euclidean norm  $k^{1/2}|\cdot|_2$ . For each  $(\eta_{ij})_{(i,j) \in \Phi} \in \Omega_k$  we define a matrix  $R(k) \in M_{Mk}(\mathbb{C})$  by

$$R(k) = \begin{bmatrix} r_{11}(k) & r_{12}(k) & \dots & r_{1M}(k) \\ 0 & r_{22}(k) & \dots & r_{2M}(k) \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & r_{MM}(k) \end{bmatrix}, \quad r_{ij}(k) = \begin{cases} y_k, & i = j \\ a_{ij}, & (i, j) \in \Theta \\ \alpha_{ij}\eta_{ij}, & (i, j) \in \Phi. \end{cases}$$

Let

$$G(k) = \text{diag}(g(k), \frac{1}{M} + g(k), \dots, \frac{M-1}{M} + g(k)) \in M_{Mk}(\mathbb{C}).$$

As a consequence of Lemma 2.4,

$$(R(k), G(k)) \in \Gamma(Z_\omega, D; m, Mk, \gamma/2).$$

Set  $\tilde{\alpha}_{ij} = \max(\alpha_{ij}, \epsilon)$  and let

$$\tilde{R}(k) = \begin{bmatrix} r_{11}(k) & r_{12}(k) & \dots & r_{1M}(k) \\ 0 & r_{22}(k) & \dots & r_{2M}(k) \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & r_{MM}(k) \end{bmatrix}, \quad r_{ij}(k) = \begin{cases} y_k, & i = j \\ a_{ij}, & (i, j) \in \Theta \\ \tilde{\alpha}_{ij}\eta_{ij}, & (i, j) \in \Phi. \end{cases} \quad (3.2)$$

Then  $\tilde{R}(k)$  lies in an  $\epsilon$ -neighborhood of  $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$ . Let  $A_l(k) \in M_{kp}(\mathbb{C})$  for  $l \in \{1, 2, \dots, N\}$  be defined by

$$A_l(k) = \begin{bmatrix} y_k & a_{f+1, f+2} & \dots & a_{f+1, f+p} \\ 0 & y_k & \dots & \vdots \\ \vdots & \ddots & \dots & a_{f+p-1, f+p} \\ 0 & \dots & 0 & y_k \end{bmatrix}$$

with  $f = (l-1)p$ . Note that we have

$$\tilde{R}(k) = \begin{bmatrix} A_1(k) & Y_{12}(k) & \dots & Y_{1N}(k) \\ 0 & A_2(k) & \dots & \vdots \\ \vdots & \dots & \ddots & Y_{N-1, N} \\ 0 & \dots & 0 & A_N(k) \end{bmatrix}, \quad (3.3)$$

where the  $Y_{ij}(k) \in M_{pk}(\mathbb{C})$  are determined by equations (3.2) and (3.3). Then, by again making use of Lemma 2.4, we have  $A_l(k) \in \Gamma_{p^2R}(\frac{1}{\sqrt{N}}T; m, pk, \gamma)$  for all  $l \in \{1, 2, \dots, N\}$ , where  $T$  is the the  $\text{DT}(\delta_0, 1)$ -operator. Let  $\epsilon > 0$  and let  $z_{k, \epsilon}$  be as

in Lemma 2.5. Let

$$B_{l,\epsilon}(k) = \begin{bmatrix} z_{k,\epsilon} & a_{f+1,f+2} & \cdots & a_{f+1,f+p} \\ 0 & z_{k,\epsilon} & \cdots & \vdots \\ \vdots & \ddots & \cdots & a_{f+p-1,f+p} \\ 0 & \cdots & 0 & z_{k,\epsilon} \end{bmatrix} \in M_{kp}(\mathbb{C}).$$

Note that the eigenvalue distribution of  $B_{l,\epsilon}(k)$  converge weakly as  $k \rightarrow +\infty$  to the measure  $\sigma_{\epsilon, \frac{1}{\sqrt{N}}}$  of Lemma 2.5.

Since every complex matrix can be put into an upper-triangular form with respect to an orthonormal basis, we can find a  $k \times k$  unitary matrix  $v(k)$  such that  $v(k)z_{k,\epsilon}v(k)^*$  is upper triangular. Since microstate spaces are invariant under conjugation by unitaries, also  $(v(k) \otimes I_M)\tilde{R}(k)(v(k) \otimes I_M)^*$  lies in an  $\epsilon$ -neighborhood of  $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$ .

For each  $1 \leq l \leq N$ , we have

$$|(v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^* - (v(k) \otimes I_p)A_l(k)(v(k) \otimes I_p)^*|_2 = |A_l(k) - B_{l,\epsilon}(k)|_2.$$

Since  $\limsup_k |B_{l,\epsilon}(k) - A_l(k)|_2 \leq \frac{\epsilon}{\sqrt{N}}$ , and taking  $N > 4$ , for  $k$  sufficiently large we have

$$|(v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^* - (v(k) \otimes I_p)A_l(k)(v(k) \otimes I_p)^*|_2 \leq \epsilon/2.$$

Set  $\tilde{B}_l(k) = (v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^*$  and  $\tilde{Y}_{ij}(k) = (v(k) \otimes I_p)Y_{ij}(k)(v(k) \otimes I_p)^*$  and denote by  $\mathcal{G}_k$  the set of all  $Mk \times Mk$  matrices of the form

$$\begin{bmatrix} \tilde{B}_1(k) & \tilde{Y}_{12}(k) & \cdots & \tilde{Y}_{1N}(k) \\ 0 & \tilde{B}_2(k) & \ddots & \cdots \\ \vdots & \ddots & \ddots & \tilde{Y}_{N-1,N}(k) \\ 0 & \cdots & 0 & \tilde{B}_N(k) \end{bmatrix},$$

over all choices of  $(\eta_{ij})_{(i,j) \in \Phi} \in \Omega_k$ . Note that the matrices in  $\mathcal{G}_k$  are upper triangular and their eigenvalue distributions are exactly the same as  $z_{k,\epsilon}$ . For  $k$  sufficiently large, the set  $\mathcal{G}_k$  lies in a  $2\epsilon$ -neighborhood of  $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$  and, therefore, in a  $2\epsilon$ -neighborhood of  $\Gamma(Z_H : D; m, Mk, \gamma)$ . Let  $\theta(\mathcal{G}_k)$  denote the unitary orbit of  $\mathcal{G}_k$  in  $M_{Mk}(\mathbb{C})$ . We will now find lower bounds for the volumes of  $\theta(\mathcal{G}_k)$  and thus, via the estimate (2.5), lower bounds for packing number of  $\Gamma(Z_H : D; m, Mk, \gamma)$ .

Denote by  $\mathcal{H}_k \subset M_{Mk}(\mathbb{C})$  the set of all matrices of the form

$$\begin{bmatrix} 0 & \tilde{Y}_{12}(k) & \cdots & \tilde{Y}_{1N}(k) \\ 0 & 0 & \ddots & \cdots \\ \vdots & \ddots & \ddots & \tilde{Y}_{N-1,N}(k) \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

over all choices of  $(\eta_{ij})_{(i,j) \in \Phi} \in \Omega_k$ . Notice that  $\mathcal{H}_k$  is isometric to the space of all matrices of the form  $(w_{ij})_{1 \leq i,j \leq M} \in M_{Mk}(\mathbb{C})$  with  $w_{ij} \in M_k(\mathbb{C})$  and

$$w_{ij} = \begin{cases} 0, & (i,j) \notin \Phi \\ \alpha_{ij}\eta_{ij}, & (i,j) \in \Phi. \end{cases}$$

It follows that  $\mathcal{H}_k$  must also have the same volume as the above subspace, computed in the ambient Hilbert space of block upper-triangular matrices with the indicated entries set to zero. Therefore,

$$\text{vol}(\mathcal{H}_k) = \text{vol}(\Omega_k) \cdot (M^{1/2})^{k^2 M(M-1)} \cdot \prod_{(i,j) \in \Phi} |\tilde{\alpha}_{ij}|^{2k^2}.$$

Let  $T_n$  the set of upper triangular matrices in  $M_n(\mathbb{C})$ ; let  $T_{n,<}$  denote the matrices in  $T_n$  that have zero diagonal, i.e. the strictly upper triangular matrices. Denote by  $\mathcal{W}_k$  the set of  $T_{Mk,<}$  consisting of all matrices  $x$  such that  $|x|_2 < \epsilon$  and  $x_{ij} = 0$  whenever  $1 \leq r < s \leq N$  and  $(r-1)pk < i \leq rpk$ ,  $(s-1)pk < j \leq spk$ . Thus,  $\mathcal{W}_k$  consists of  $N \times N$  diagonal matrices whose diagonal entries are strictly upper triangular  $pk \times pk$  matrices. Denote by  $\mathcal{D}_k$  the subset of diagonal matrices  $x$  of  $M_{Mk}(\mathbb{C})$  such that  $|x|_2 < \epsilon$ . It follows that if  $f_k$  is the matrix

$$f_k = \begin{bmatrix} \tilde{B}_1(k) & 0 & \dots & 0 \\ 0 & \tilde{B}_2(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \tilde{B}_N(k) \end{bmatrix}$$

then  $f_k + \mathcal{D}_k + \mathcal{W}_k + \mathcal{H}_k \subset \mathcal{N}_{3\epsilon}(\mathcal{G}_k)$ , where the  $3\epsilon$ -neighborhood is taken in the ambient space  $T_{Mk}$  with respect to the metric induced by  $|\cdot|_2$ . Now observe that the space of diagonal  $Mk \times Mk$  and  $T_{Mk,<}$  are orthogonal subspaces. Let  $\theta_{3\epsilon}(\mathcal{G}_k)$  denote the  $3\epsilon$  neighborhood of the unitary orbit of  $\theta(\mathcal{G}_k)$  of  $\mathcal{G}_k$ . Let  $dX$  denote Lebesgue measure on  $T_{Mk}$  corresponding to the Euclidean norm  $(Mk)^{1/2}|\cdot|_2$ , which is coordinatized by the complex entries  $X = \{x_{ij}\}_{1 \leq i \leq j \leq Mk}$  of the matrix. Using Dyson's formula we have

$$\begin{aligned} \text{vol}(\theta_{3\epsilon}(\mathcal{G}_k)) &\geq C_{Mk} \cdot \int_{f_k + \mathcal{D}_k + \mathcal{W}_k + \mathcal{H}_k} \prod_{1 \leq i < j \leq Mk} |x_{ii} - x_{jj}|^2 dX \\ &= C_{Mk} \cdot \text{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot \int_{D(f_k + \mathcal{D}_k)} \prod_{1 \leq i < j \leq Mk} |x_{ii} - x_{jj}|^2 dx_{11} \cdots dx_{Mk,Mk} \\ &\geq C_{Mk} \cdot \text{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot E_\epsilon(f_k) \end{aligned} \tag{3.4}$$

where the constant  $C_{Mk}$  is as in [6] and where  $\text{vol}(\theta_{3\epsilon}(\mathcal{G}_k))$  is computed in  $M_{Mk}(\mathbb{C})$  and  $\mathcal{W}_k + \mathcal{H}_k$  is computed in  $T_{Mk,<}$ , both being Euclidean volumes corresponding to the norms  $(Mk)^{1/2}|\cdot|_2$ , where the integral over  $D(f_k + \mathcal{D}_k)$  is over the diagonal parts of these matrices, and where  $E_\epsilon(f_k)$  is the integral defined on p. 252 of [6]. It is clear that  $\theta_{3\epsilon}(\mathcal{G}_k) \subset \mathcal{N}_{4\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma))$ , so (3.4) gives a lower bound on  $\text{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma)))$ .

Using (3.4) and the standard volume comparison test (2.5), we have

$$\begin{aligned} P_{2\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma)) &\geq \frac{\text{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z_H; m, Mk, \gamma)))}{\text{vol}(\mathcal{B}_{8\epsilon})} \\ &\geq C_{Mk} \cdot \text{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot E_\epsilon(f_k) \cdot \frac{\Gamma((Mk)^2 + 1)}{\pi^{(Mk)^2} (8(Mk)^{1/2}\epsilon)^{2(Mk)^2}} \end{aligned}$$

where  $\mathcal{B}_{8\epsilon}$  is a ball in  $M_{Mk}(\mathbb{C})$  of radius  $8\epsilon$  with respect to  $|\cdot|_2$ , and we are taking volumes corresponding to the Euclidean norm  $(Mk)^{1/2}|\cdot|_2$ . Since  $\mathcal{W}_k$  and  $\mathcal{H}_k$  are orthogonal, we have that  $\text{vol}(\mathcal{W}_k + \mathcal{H}_k) = \text{vol}(\mathcal{W}_k) \cdot \text{vol}(\mathcal{H}_k)$ , where each volume is taken in the subspace of appropriate dimension. But  $\mathcal{W}_k$  is a ball of radius  $(Mk)^{1/2}\epsilon$  in a space of real dimension  $Npk(pk-1)$ , so

$$\text{vol}(\mathcal{W}_k + \mathcal{H}_k) = \frac{\pi^{\frac{Npk(pk-1)}{2}} ((Mk)^{1/2}\epsilon)^{Npk(pk-1)}}{\Gamma(\frac{Npk(pk-1)}{2} + 1)} \cdot \text{vol}(\mathcal{H}_k)$$

where  $\text{vol}(\mathcal{H}_k) = \text{vol}(\Omega_k) \cdot (M^{1/2})^{k^2 M(M-1)} \cdot \prod_{(i,j) \in \Phi} |\tilde{\alpha}_{ij}|^{2k^2}$ . Using Stirling's formula and  $M = Np$ , we find

$$\begin{aligned} \mathbb{P}_\epsilon(Z_H : D; m, \gamma) &\geq \liminf_k (Mk)^{-2} \log P_\epsilon(\Gamma(Z_H : D; m, Mk, \gamma)) \\ &\geq \liminf_k (Mk)^{-2} \log(E_\epsilon(f_k)) \\ &\quad + \liminf_k \left( (Mk)^{-2} \log(C_{Mk}) + (Mk)^{-2} \log(\text{vol}(\Omega_k)) + \right. \\ &\quad + \left( 2 - \frac{1}{N} \right) |\log \epsilon| + \left( 1 - \frac{1}{2N} \right) \log k \\ &\quad + \left. \left( \frac{M-1}{2M} \right) \log M + \frac{2}{M^2} \sum_{(i,j) \in \Phi} \log |\tilde{\alpha}_{ij}| \right) + L_1 \\ &= \liminf_k (Mk)^{-2} \log(E_\epsilon(f_k)) \\ &\quad + \liminf_k \left( (Mk)^{-2} \log C_{Mk} + \frac{1}{2} \log Mk \right) \\ &\quad + \liminf_k \left( (Mk)^{-2} \log(\text{vol}(\Omega_k)) + \left( \frac{1}{2} - \frac{1}{2N} \right) \log k \right) \\ &\quad + \left( 2 - \frac{1}{N} \right) |\log \epsilon| + \frac{2}{M^2} \sum_{(i,j) \in \Phi} \log |\tilde{\alpha}_{ij}| + L_2 \end{aligned}$$

where  $L_1$  and  $L_2$  are constants independent of  $\epsilon, m$  and  $\gamma$ . As  $\gamma \rightarrow 0$  and  $m \rightarrow +\infty$ , we have convergence

$$\frac{2}{M^2} \sum_{(i,j) \in \Phi} \log |\tilde{\alpha}_{ij}| \longrightarrow 2 \iint_{K_N} \log(\max(H(s, t), \epsilon)) ds dt$$

where

$$K_N = \bigcup_{j=1}^{N-1} \left\{ \frac{j}{N} \leq x \leq \frac{j+1}{N} \leq y \leq 1 \right\}.$$

Note that we have  $\text{area}(K_N) = \frac{N(N-1)}{2N^2}$ . Now by (3.1), we have

$$\begin{aligned} \liminf_k \left( (Mk)^{-2} \log(\text{vol}(\Omega_k)) + \left( \frac{1}{2} - \frac{1}{2N} \right) \cdot \log(k) \right) \\ \geq M^{-2} \chi \left( \{\text{Re} C_{ij}\}, \{\text{Im} C_{ij}\} : (i, j) \in \Phi \right) \end{aligned}$$

Then

$$\begin{aligned} \mathbb{P}_\epsilon(Z_H : D) &\geq \liminf_k (Mk)^{-2} \log(E_\epsilon(f_k)) + \left( 2 - \frac{1}{N} \right) |\log \epsilon| \\ &\quad + 2 \iint_{K_N} \log(\max(H(s, t), \epsilon)) ds dt + L_3 \end{aligned}$$

The eigenvalue distribution of  $f_k$  equals that of  $z_{k,\epsilon}$  and converges as  $k \rightarrow +\infty$  to the measure  $\sigma_{\epsilon, \frac{1}{\sqrt{N}}}$ , we may apply Lemma 2.3 of [6] concerning the asymptotics of  $E_\epsilon(f_k)$  as  $k \rightarrow \infty$ . Using also Lemma 3.1, we get

$$\delta_0(Z_H : D) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(Z_H : D)}{|\log \epsilon|} \geq 1 + 2 \cdot \text{area}(\text{supp}(H) \cap K_N).$$

Taking  $N$  arbitrarily large completes the proof.  $\square$

The following Theorem gives us an upper bound on  $\delta_0(Z_H : D)$  without any conditions on the support of  $H$ .

**Theorem 3.3.** *Let  $H \geq 0$ ,  $H \in L^1([0, 1]^2)$  have essentially bounded coordinate expectations  $CE_1(H)$  and  $CE_2(H)$ , as in equations (2.1). Then*

$$\delta_0(Z_H : D) \leq \min\{2, 1 + 2 \text{area}(\text{supp}(H))\}.$$

*Proof.* First of all it is clear that  $\delta_0(Z_H : D) \leq \delta_0(Z_H) \leq 2$ .

By standard arguments we can find  $\omega$  in regular block form such that both  $\|Z_H - Z_\omega\|_2$  and  $\text{area}(\text{supp}(H) \Delta \text{supp}(\omega))$  are arbitrarily small. Using this, given  $\delta > 0$  we can find projections  $p_1, q_1, p_2, q_2, \dots, p_n, q_n$  in  $W^*(D)$  such that if  $i \neq j$ , then  $p_i \otimes q_i$  is orthogonal to  $p_j \otimes q_j$  in  $W^*(D) \overline{\otimes} W^*(D)$  and such that

$$\sum_{i=1}^n \tau(p_i) \tau(q_i) > 1 - \text{area}(\text{supp}(H)) - \delta/3 \quad (3.5)$$

$$\sum_{i=1}^n \|p_i Z_H q_i\|_2 < \delta/4. \quad (3.6)$$

Take  $R > \max\{\|Z_H\|_2, \|D\|_2\}$ . Using Lemma 2.9 of [13], given  $\epsilon > 0$  there exist  $m_0, \gamma_0, k_0$  such that for  $m \geq m_0$ ,  $\gamma < \gamma_0$ ,  $k \geq k_0$  and for every  $(A, B)$  and  $(\tilde{A}, \tilde{B}) \in \Gamma_R(Z_H, D; m, k, \gamma)$  there exists a unitary  $U \in M_k(\mathbb{C})$  such that

$$\|U \tilde{B} U^* - B\|_2 < \epsilon. \quad (3.7)$$

For  $m$  and  $k$  sufficiently big and  $\gamma$  sufficiently small we can find spectral projections of  $B$

$$P_1, Q_1, \dots, P_n, Q_n \in M_k(\mathbb{C})$$

and spectral projections of  $\tilde{B}$

$$\tilde{P}_1, \tilde{Q}_1, \dots, \tilde{P}_n, \tilde{Q}_n \in M_k(\mathbb{C})$$

such that if  $i \neq j$  then  $P_i \otimes Q_i$  is orthogonal to  $P_j \otimes Q_j$  in  $M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$  and  $\tilde{P}_i \otimes \tilde{Q}_i$  is orthogonal to  $\tilde{P}_j \otimes \tilde{Q}_j$  satisfying

$$\begin{aligned} |\mathrm{tr}_k(P_i) - \tau(p_i)| &< \frac{\delta}{3n}, \quad |\mathrm{tr}_k(Q_i) - \tau(q_i)| < \frac{\delta}{3n}, \quad \sum_{i=1}^n \|P_i A Q_i\|_2 < \frac{\delta}{2} \\ |\mathrm{tr}_k(\tilde{P}_i) - \tau(p_i)| &< \frac{\delta}{3n}, \quad |\mathrm{tr}_k(\tilde{Q}_i) - \tau(q_i)| < \frac{\delta}{3n}, \quad \sum_{i=1}^n \|\tilde{P}_i \tilde{A} \tilde{Q}_i\|_2 < \frac{\delta}{2}. \end{aligned}$$

Taking  $\epsilon$  sufficiently small and using (3.7) together with the fact that we can always approximate these projections with polynomials in  $B$  and  $\tilde{B}$  in the  $\|\cdot\|_2$ , we can also guarantee that

$$\|P_i - U \tilde{P}_i U^*\|_2 < \frac{\delta}{6nR}, \quad \|Q_i - U \tilde{Q}_i U^*\|_2 < \frac{\delta}{6nR} \quad (1 \leq i \leq n).$$

Therefore,

$$\sum_{i=1}^n \|P_i(U \tilde{A} U^*) Q_i\|_2 < \sum_{i=1}^n \left( \frac{3\delta \|\tilde{A}\|}{6nR} + \|\tilde{P}_i \tilde{A} \tilde{Q}_i\|_2 \right) < \delta. \quad (3.8)$$

Let  $\Omega_R(H, k) = \{X \in M_k(\mathbb{C}) : \|X\|_2 \leq R, P_i X Q_i = 0 \text{ for } i = 1, \dots, n\}$ , this is a ball of radius  $R$  in a space of real dimension  $d(k) = 2k^2(1 - \sum_{i=1}^n \mathrm{tr}_k(P_i) \mathrm{tr}_k(Q_i))$ . By (3.8) it is clear that

$$\Gamma_R(Z_H : D; m, k, \gamma) \subseteq \theta(N_\delta(\Omega_R(H, k))) \quad (3.9)$$

where  $\theta(N_\delta(\Omega_R(H, k)))$  is the unitary orbit of the  $\delta$ -neighborhood of  $\Omega_R(H, k)$ . Taking the  $P_\delta$  packing number on both sides of (3.9), we get

$$P_\delta(\Gamma_R(Z_H : D; m, k, \gamma)) \leq P_\delta(\theta(N_\delta(\Omega_R(H, k)))) \leq P_\delta(U_k(\mathbb{C})) \cdot P_\delta(N_\delta(\Omega_R(H, k))).$$

Using Theorem 7 of [18], there exists a constant  $K_1$  independent of  $k$  such that

$$P_\delta(U_k(\mathbb{C})) \leq \left( \frac{K_1}{\delta} \right)^{k^2}. \quad (3.10)$$

On the other hand, standard packing number estimations gives us

$$P_\delta(N_\delta(\Omega_R(H, k))) \leq P_\delta(\Omega_{R+\delta}(H, k)) \leq \left( \frac{K_2(R+\delta)}{\delta} \right)^{d(k)} \quad (3.11)$$

where  $K_2$  is a constant independent of  $k$ . It follows that

$$P_\delta(\Gamma_R(Z_H : D; m, k, \gamma)) \leq \left( \frac{K_1}{\delta} \right)^{k^2} \cdot \left( \frac{K_2(R+\delta)}{\delta} \right)^{d(k)}.$$

Now using (3.5) yields

$$\begin{aligned} \frac{d(k)}{k^2} &= 2 \left( 1 - \sum_{i=1}^n \text{tr}_k(P_i) \text{tr}_k(Q_i) \right) \leq 2 \left( 1 - \sum_{i=1}^n \tau(p_i) \tau(q_i) + 2\delta/3 \right) \\ &\leq 2 \left( \text{area}(\text{supp}(H)) + \delta \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_k \frac{1}{k^2} \log(P_\delta(\Gamma_R(Z_H : D; m, k, \gamma))) &\leq \log(K_1) + |\log(\delta)| + \\ &+ 2(\text{area}(\text{supp}(H)) + \delta) \cdot \log(K_2(R + \delta)) \\ &+ 2(\text{area}(\text{supp}(H)) + \delta) \cdot |\log(\delta)|. \end{aligned}$$

When  $\gamma \rightarrow 0$  and  $m \rightarrow +\infty$  we obtain

$$\mathbb{P}_\delta(Z_H : D) \leq (1 + 2 \cdot \text{area}(\text{supp}(H)) + 2\delta) \cdot |\log \delta| + C$$

where  $C$  is a constant. It follows that

$$\delta_0(Z_H : D) = \limsup_{\delta \rightarrow 0} \frac{\mathbb{P}_\delta(Z_H : D)}{|\log \delta|} \leq 1 + 2 \cdot \text{area}(\text{supp}(H)).$$

□

#### 4. CONCLUDING REMARKS AND QUESTIONS

Since the free entropy dimension of  $Z_H$  in the presence of  $D$  is a lower bound for the free entropy dimension of  $Z_H$ , from Theorems 3.2 and 3.3 we have that for any  $H$  as in Theorem 3.2,

$$1 + 2 \text{area}(\text{supp}(H)) = \delta_0(Z_H : D) \leq \delta_0(Z_H). \quad (4.1)$$

However,  $1 + 2 \text{area}(\text{supp}(H))$  is not the actual value of  $\delta_0(Z_H)$  in all cases. For example, if  $n \geq 2$  and if  $H$  is the characteristic function of  $\cup_{i=1}^n T_i$ , where  $T_i = \{(x, y) \in [0, 1] : \frac{i-1}{n} \leq x < y \leq \frac{i}{n}\}$ , then the moments of  $Z_H$  agree with the moments of a nonzero multiple of the quasinilpotent DT-operator  $T$ . Therefore, in this case we have

$$\delta_0(Z_H : D) = 1 + \frac{1}{n} < \delta_0(Z_H) = \delta_0(T) = 2. \quad (4.2)$$

Of course, if  $D$  belongs to the von Neumann algebra generated by  $Z_H$ , then equality holds in (4.1). It is an interesting question, when do we have  $D \in W^*(\{Z_H\})$ ? More generally, what is the von Neumann algebra generated by  $Z_H$ ? When is it a factor? Is it then an interpolated free group factor? A particular case of interest is when  $H$  is the characteristic function of the band

$$\{(x, y) \mid 0 \leq x < y < \min(1, x + \alpha)\},$$

for  $\alpha \in (0, 1)$ , as is drawn in Figure 2 (on page 3).



## REFERENCES

- [1] Aagard L. and Haagerup U., *Moment formulas for the quasi-nilpotent DT-operator*, Int. J. Math 15 (2004), 581-628.
- [2] Dykema K., *Free products of hyperfinite von Neumann algebras*, Duke Math. J. 69 (1993), 97-119.
- [3] Dykema K., *Hyperinvariant subspaces for some B-circular operators*, with an appendix by Gabriel Tucci, Math. Ann. 333 (2005), 485-523.
- [4] Dykema K. and Haagerup U., *DT-operators and decomposability of Voiculescu's circular operator*, Amer. J. Math. 126 (2004), 121-289.
- [5] Dykema K. and Haagerup U., *Invariant subspaces of the quasinilpotent DT-operator*, J. Funct. Anal. 209 (2004), 332-366.
- [6] Dykema K., Jung K. and Shlyakhtenko D., *The microstates free entropy dimension of any DT-operator is 2*, Documenta Math. 10 (2005), 247-261.
- [7] Ge L. and Shen J., *On the free entropy dimension of finite von Neumann algebras*, Geom. Funct. Anal. 12 (2002), 546-566.
- [8] Jung K., *The free entropy dimension of hyperfinite von Neumann algebras*, Trans. Amer. Math. Soc. 355 (2003), 5053-5089.
- [9] Jung K., *A free entropy dimension lemma*, Pacific J. Math. 177 (2003), 265-271.
- [10] Jung K., *A hyperfinite inequality for free entropy dimension*, Proc. Amer. Math. Soc. 134 (2006), 2099-2108.
- [11] Jung K., *Strongly 1-bounded von Neumann algebras*, Geom. Funct. Anal. (to appear), preprint arXiv:math.OA/0510576 v2.
- [12] Jung K., *Some free entropy dimension inequalities for subfactors*, preprint arXiv:math.OA/0410594.
- [13] Jung K., *Amenability, tubularity, and embeddings into  $\mathcal{R}^\omega$* , preprint arXiv:math.OA/0506108 v2.
- [14] Jung K., Shlyakhtenko D., *All generating sets of all property T von Neumann algebras have free entropy dimension  $\leq 1$* , preprint arXiv: math.OA/0603669.
- [15] Mehta M.L., *Random Matrices, second edition*, Academic Press, 1991.
- [16] Radulescu F., *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index*, Invent. Math. 115 (1994), 347-389.
- [17] Śniady P., *Random regularization of Brown measure*, J. Funct. Anal. 193 (2002), 291-313.
- [18] Szarek S., *Metric Entropy of homogeneous spaces*, Quantum Probability, (Gdansk 1977), Banach Center Publications v.43, Polish Academy of Science, Warsaw 1998, 395-410.
- [19] Voiculescu D., *Circular and semicircular systems and free product factors*, Operator Algebras, Unitary Representations, Algebras, and Invariant Theory, Progress in Mathematics **92**, Birkhäuser, 1990, pp. 45-60.
- [20] Voiculescu D., *Limit laws for random matrices and free products*, Invent. Math. 104 (1991), 201-220.
- [21] Voiculescu D., *The analogues of entropy and of Fisher's information measure in the free probability theory, II*, Invent. Math. 118 (1994), 411-440.
- [22] Voiculescu D., *The analogues of entropy and of Fisher's information measure in the free probability theory III: The absence of Cartan subalgebras*, Geom. Funct. Anal. 6 (1996), 172-199.
- [23] Voiculescu D., *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Internat. Math. Res. Notices (1998), 41-64.
- [24] Voiculescu D., *Free entropy dimension  $\leq 1$  for some generators of property T factors of type  $\text{II}_1$* , J. reine angew. Math. 514 (1999), 113-118.
- [25] Voiculescu D., *Free entropy*, Bull. London Math. Soc. 34 (2002), 257-332.
- [26] von Neumann J., *Approximative properties of matrices of high finite order*, Portugal. Math. 3 (1942), 1-62.

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